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Infinitesimal Hilbert 16th Problem

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Hilbert 16th Problem (Second Part) is the following:
Given

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}, \tag{1}$$

where P and Q are real polynomials of x and y, what may be said about the number of limit cycles and their relative positions.

Infinitesimal Hilbert 16th problem is one of the restricted version of Hilbert 16th problem, it was posted by V. I. Arnold in 1977 [1] as follows:
Given

$$H(x,y) = h, \quad \omega = Q_n dx - P_n dy,$$

where H, P_n and Q_n are real polynomials of x and y,

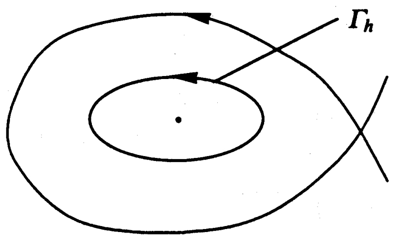

Fig. 1

$\max(\deg P_n, \deg Q_n) = n$, $\deg H = m + 1$. Then

$$\Gamma_h \subset (H = h),$$
$$I(h) = \int_{\Gamma_n} \omega.$$

Problem: What is the number of isolated zeros of $I(h)$?

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This problem is closely related to the problem of estimating the number of limit cycles of the following Hamiltonian System with small perturbations:

$$\begin{aligned} \frac{dx}{dt} &= \frac{\partial H}{\partial y} + \varepsilon P_n, \\ \frac{dy}{dt} &= -\frac{\partial H}{\partial x} + \varepsilon Q_n, \end{aligned} \tag{2}$$

where $0 < |\varepsilon| \ll 1$.

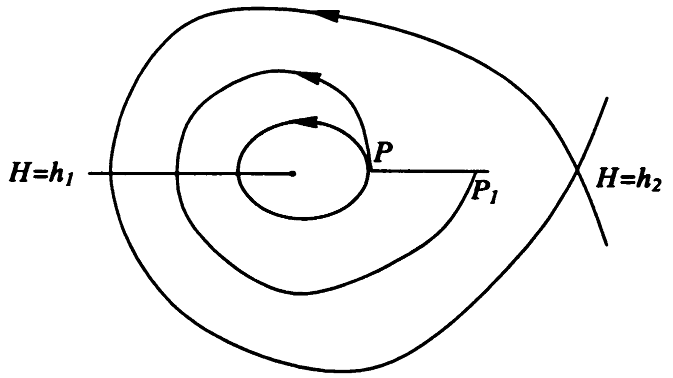

Fig. 2

Then we have

$$\begin{aligned} \Delta H &= \int_0^{T(P,\varepsilon)} \frac{dH}{dt}\Big|_{(2)} dt \\ &= \varepsilon \int_0^{T(P,\varepsilon)} \left(\frac{\partial H}{\partial x} P_n + \frac{\partial H}{\partial y} Q_n \right) dt \\ &= \varepsilon \int_{\Gamma_h} (Q_n dx - P_n dy) + O(\varepsilon^2) \\ &= \varepsilon I(h) + O(\varepsilon^2), \quad h_1 < h < h_2, \end{aligned}$$

where $T(P,\, \varepsilon)$ is the time along $\widehat{PP_1}$. Hence, there exists a periodic orbit of (2) passing through $P \in \Gamma_h$ if and only if $\Delta H(h) = 0$, i.e. $I(h)$ is the first approximation of the Poincarè return map ΔH with respect to "ε".

Relationship:

(i) If $I(h) = 0, I'(h) \neq 0 \implies \exists\, hyp.$ limt cycle $L_{h(\varepsilon)} \to \Gamma_h$ $(\varepsilon \to 0)$; conversely, if $\exists\, hyp.$ limt cycle $L_{h(\varepsilon)} \to \Gamma_h$ $(\varepsilon \to 0) \implies I(h) = 0$.

(ii) If $I(h) = I'(h) = \cdots = I^{(k-1)}(h) = 0,\ I^{(k)}(h) \neq 0 \implies$ there are at most k limit cycles bifurcated from L_h.

(iii) $\#\{$limit cycle $s.t.\ L_{h(\varepsilon)} \to \Gamma_h\ (\varepsilon \to 0)\} \leq \#\{I(h) = 0\} = Z(m,n)$.

Here multiplicity taking into account.

In this talk it will be introduced the main results and main methods of how to stimate the number of isolated zeros of $I(h)$.

The main methods are as follows:

1. Argument principle [2], [3];
2. Generalized Rolle Theorem [4], [5];
3. Estimate zeros of analytic functions [6], [7].

References

[1] V. I. Arnold, Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields, *Funct. Anal. Appl.* **11** (1977), 85–92.

[2] G. S. Petrov, Number of zeros of complete elliptic integrals, *Funct. Anal. Appl.* **18** (1984), 148–149.

[3] Baoyi Li and Zhifen Zhang, A note on a result of G. S. Petrov about the weakened 16th Hilbert problem, *J. Math. Anal. Appl.* **190** (1995), 489–516.

[4] E. Horozov and I. D. Iliev, Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians, *Nonlinearity* **11** (1998), 1521–1537.

[5] Yulin Zhao and Zhifen Zhang, Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonians, *J. Differential Equations* **155** (1999), 73–88.

[6] Yu. S. Il'yashenko and S. Yakovenko, Double exponential estimate for the number of zeros of complete Abelian integrals, *Invent. Math.* **121** (1995), 613–650.

[7] D. Novikov and S. Yakovenko, Simple exponential estimate for the number of real zeros of complete Abelian integrals, *Ann. Inst. Fourier (Grenoble)* **45** (1995), 897–927.